1 a) S^2 is nonempty because $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in S^2$ is symmetric.

Let $A, B \in S^2$. Then $A = A^T$ and $B = B^T$.

Hence, $(A+B)^T = A^T + B^T = A + B$.

So S^2 is closed under vector addition.

Finally, let $A \in S^2$ and $c \in \mathbb{R}$.

Then $(cA)^T = cA^T = cA$.

We conclude that S^2 is closed under scalar multiplication. Thus, S^2 is a subspace of $\mathbb{R}^{2\times2}$.

b) Let $\begin{bmatrix} a & b \\ b & c \end{bmatrix} \in S^2$.

Then $\begin{bmatrix} a & b \\ b & c \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

So $S^2 \subseteq \operatorname{span}\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)$

Now let $a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in \operatorname{span}\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)$.

Note that $a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \in S^2$. So $S^2 = \operatorname{span}\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)$.

It remains to be shown that these matrices are linearly independent. Let $c_1, c_2, c_3 \in \mathbb{R}$ be such that

$$c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 0$$

Then $\begin{bmatrix} c_1 & c_3 \\ c_3 & c_2 \end{bmatrix} = 0 \implies c_1 = c_2 = c_3 = 0.$

So $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ are linearly independent.
We conclude that

$$\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right)$$

is a basis of $\mathcal{S}^2$.

c) Let $\alpha, \beta \in \mathbb{R}$ and $P, Q \in \mathcal{S}^2$.

Then $L_A(\alpha P + \beta Q) = \alpha P + \beta Q - A^T(\alpha P + \beta Q)A$
$= \alpha(P - A^T P A) + \beta(Q - A^T Q A)$
$= \alpha L_A(P) + \beta L_A(Q).$

So indeed, L_A is a linear operator.

d) Let $A = I_2$. Then $P - A^T P A = P - I \cdot P \cdot I$
$= 0$ for all $P \in S^2$. So $\ker L_I = S^2 \neq \{0\}$.

e) Let $P \in \ker L_A$. $\Rightarrow$ $P - A^T P A = 0$.
$\Rightarrow$ $(A^T)^k (P - A^T P A) A^k = 0$ for all $k = 0, 1, \dots$

$\Rightarrow$ $(A^T)^k P A^k = (A^T)^{k+1} P A^{k+1}$ for all $k = 0, 1, \dots$

$\Rightarrow$ $P = A^T P A = (A^T)^2 P A^2 = \dots = (A^T)^k P A^k \; \forall k = 0, 1, \dots$

$$\Rightarrow P = \lim_{k \to \infty} (A^T)^k P A^k = 0$$

So $\ker L_A = \{0\}$.

f) Define $P := \sum_{k=0}^{\infty} (A^T)^k Q A^k$.

Then $P - A^T P A =$

$$\sum_{k=0}^{\infty} (A^T)^k Q A^k - \sum_{k=1}^{\infty} (A^T)^k Q A^k = Q$$

so P is a solution. Let $\bar{P}$ be an arbitrary solution to $L_A(\bar{P}) = Q$.

Then, since L_A is linear, $L_A(P-\bar{P}) = Q - Q = 0$

So $P-\bar{P} \in \ker L_A \Rightarrow P = \bar{P}$ by e).

So P is the unique solution to $L_A(P) = Q$.

g) $$L_A\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} a_{11}^2 & a_{11}a_{12} \\ a_{11}a_{12} & a_{12}^2 \end{bmatrix}$$

$$= (1-a_{11}^2)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - a_{12}^2\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - a_{11}a_{12}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$L_A\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ a_{21} & a_{22} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} a_{21}^2 & a_{22}a_{21} \\ a_{22}a_{21} & a_{22}^2 \end{bmatrix}$$

$$= -a_{21}^2\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (1-a_{22}^2)\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - a_{22}a_{21}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$L_A\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}\begin{bmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 2a_{11}a_{21} & a_{11}a_{22} + a_{12}a_{21} \\ a_{12}a_{21} + a_{11}a_{22} & 2a_{12}a_{22} \end{bmatrix}$$

$$= -2a_{11}a_{21}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - 2a_{12}a_{22}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + (1 - a_{12}a_{21} - a_{11}a_{22})\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

So the matrix representation of L_A is:

$$\begin{pmatrix} 1-a_{11}^2 & -a_{21}^2 & -2a_{11}a_{21} \\ -a_{12}^2 & 1-a_{22}^2 & -2a_{12}a_{12} \\ -a_{11}a_{12} & -a_{22}a_{21} & 1-a_{11}a_{22}-a_{12}a_{21} \end{pmatrix}$$

$$= I_3 - \begin{pmatrix} a_{11}^2 & a_{21}^2 & 2a_{11}a_{21} \\ a_{12}^2 & a_{22}^2 & 2a_{12}a_{12} \\ a_{11}a_{12} & a_{22}a_{21} & a_{11}a_{22}+a_{12}a_{21} \end{pmatrix}.$$